MAΘ Competition Team Homework Set 4

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Due date: November 7, 2025

Problem 1. Wilhelmina the Witch gives out 3 candies to each trick-or-treater who knocks. Prove with induction that after n trick-or-treaters knock, she has given out $3n$ pieces of candy.

Problem 2. Dr. Frank N. Stein is creates a monster on the eve of Halloween. Each subsequent day, he creates double the amount of monsters. He thinks he can express the total number of monsters he creates n days since Halloween as

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1.$$

Prove that this expression is valid using induction.

Problem 3. Milhouse the Mummy is building a pyramid for his pharaoh. He starts with a square base of n cubits long, where n is an odd number, and builds up, with each subsequent base being 2 cubits shorter, until he finishes with the 1-cubit-long capstone. Use induction to prove the expression for the total number of cube-cubits in his pyramid is

$$\frac{n(4n^2 - 1)}{3}$$

Problem 4. Every night Count Dracula's power level doubles, but Van Helsing's power only grows quadratically. Dracula boasts that from the fifth day onwards, his power will always exceed Van Helsing's. Show he is right by using induction to prove that

$$2^n > n^2, \text{ for } n \geq 5, n \in \mathbb{N}$$

Problem 5. Oh no! A weirdly math-obsessed zombie has taken your family hostage, and only promises to let them go if you can prove, using induction, De Moivre's theorem, which states that:

$$(\cos\theta + i\sin\theta)^n \equiv \cos n\theta + i\sin n\theta, n \in \mathbb{N}.$$

Can you save your family? (Hint: remember the angle addition formulas for sin and cos!)

Problem 6. Challenge He's done it! Gary the Ghost proclaims that he has proved that all ghosts are the same color using induction. "For one ghost, this is obviously true," he says. "Now suppose any group of k ghosts are the same color. Consider a group of $k+1$ ghosts! The first k ghosts are the same color by the inductive hypothesis, and the last k ghosts are also the same color because we assume that a group of k ghost always has the same color! Since these groups overlap, $k+1$ ghosts must have the same color too!" What's wrong with Gary the Ghost's proof?