MAΘ Competition Team HW Set 9

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Problem 1. An 8×8 chessboard is colored in the usual way, but that's boring, so you decide to fix this. You can take any row, column, or 2×2 square, and reverse the colors inside it, switching black to white and white to black. Prove that it's impossible to end up with 63 white squares and 1 black square.

Problem 2. The numbers 1, 2, . . . , 100 are written on a blackboard. You may choose any two numbers a and b and erase them, replacing them with the single number $a + b - 1$. After 99 steps, only a single number will be left. What is it?

Problem 3. A room is initially empty. Every minute, either two people enter or one person leaves. After exactly 3^{3^3} minutes, could the room contain exactly $3^{3^3} + 1$ people?

Problem 4. There are $a + b$ bowls arranged in a row, numbered 1 through $a + b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear.

A legal move consists of moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j - 1$, provided that the difference $i - j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

Problem 5. Let $P_1, \ldots, P_{2n}$ be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$ other than $(1, 0)$. Each point is colored either red or blue, with exactly n of them red and exactly n of them blue. Let $R_1, \ldots, R_n$ be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 traveling counterclockwise around the circle from R_2, and so on, until we have labeled all the blue points $B_1, \ldots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \ldots, R_n$ of the red points.

Problem 6(Challenge). To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.